

INTEGRABLE REDUCTIONS OF ANTI-SELF-DUAL YANG-MILLS EQUATION THAT PRESERVE THE SOLUTION STRUCTURE

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INTRODUCTION

The anti-self-dual Yang-Mills (ASDYM) equation is an important model in quantum field theory, twistor theory and several areas in mathematical physics. In the present research, we consider the following $SL(2)$ ASDYM hierarchies ($n, m \in \mathbb{Z}$):

- **The ASDYM hierarchy** ($\mathbf{x} := (\cdots, x_{-1}, x_0, x_1, \cdots)$):

$$(\mathbf{V}_{x_{m+1}} \mathbf{V}^{-1})_{x_n} - (\mathbf{V}_{x_{n+1}} \mathbf{V}^{-1})_{x_m} = 0, \quad \mathbf{V}(\mathbf{x}) \in SL(2). \quad (1)$$

- **The potential ASDYM hierarchy:**

$$\mathbf{U}_{x_n, x_{m+1}} - \mathbf{U}_{x_m, x_{n+1}} - [\mathbf{U}_{x_n}, \mathbf{U}_{x_m}] = 0, \quad \mathbf{U}(\mathbf{x}) \in \mathfrak{sl}(2). \quad (2)$$

- **The Miura transformation:**

$$\mathbf{U}_{x_n} = -\mathbf{V}_{x_{n+1}} \mathbf{V}^{-1}, \quad \mathbf{U}_{x_m} = -\mathbf{V}_{x_{m+1}} \mathbf{V}^{-1}. \quad (3)$$

The ASDYM equation allows reductions towards lower-dimensional integrable systems, e.g.: the KdV, NLS, sG equations (AKNS system), the KP equation, the Painlevé equations etc. On the basis of these facts, R.S. Ward proposed a conjecture [1]:

... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.

Since then, there raises a trend of searching such kinds of integrable reductions by the implement of gauge potential functions (GPF) [2].

Motivation

A recent research [3] established the reduction from (3) to the AKNS(-1) and pKN(-1) system. By using

$$\mathbf{V}_{x_1} \mathbf{V}^{-1} = -[\mathbf{U}, \mathbf{a}], \quad [\mathbf{V}, \mathbf{a}] \mathbf{V}^{-1} = -\mathbf{U}_{x_1}, \quad \mathbf{a} = \text{diag}(1, 0),$$

we found $(q, r) := (U_{21}, U_{12})$ and $(q, v) := (U_{21}, iV_{12}/V_{22})$ solve the following systems respectively:

- **The AKNS(-1) system (sine-Gordon equation):**

$$q_{x_1, x_{-1}} - q + 2q\partial_{x_1}^{-1}(qr)_{x_{-1}} = 0, \quad (4a)$$

$$r_{x_1, x_{-1}} - r + 2r\partial_{x_1}^{-1}(qr)_{x_{-1}} = 0. \quad (4b)$$

- **The pKN(-1) system (Fokas-Lenells equation):**

$$q_{x_1, x_{-1}} - q - 2iqvq_{x_{-1}} = 0, \quad (5a)$$

$$v_{x_1, x_{-1}} - v + 2ivqv_{x_{-1}} = 0. \quad (5b)$$

Considering the ASDYM solutions in [4,5] (derived from Cauchy matrix approach), we naturally obtain the solutions of sG and FL equations respectively. This result indicates two facts:

- **The ASDYM can be reduced to the FL equation.**

- **The reduction process preserve the solution structure.**

There naturally rises a problem that whether this result can be extended to the whole hierarchy, i.e. can we reduce (3) to the AKNS and KN hierarchies without changing the solution structure?

REFERENCES

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Integrable Hierarchies

- **The AKNS hierarchy** ($\partial^{-1} = \partial_x^{-1}$):

$$q_{t_{n+1}} = q_{x, t_n} + 2q\partial^{-1}(qr)_{t_n}, \quad (6a)$$

$$r_{t_{n+1}} = -r_{x, t_n} - 2r\partial^{-1}(rq)_{t_n}, \quad (6b)$$

- **The KN hierarchy** ($\bar{\partial}^{-1} = \partial_{\bar{x}}^{-1}$):

$$u_{\tau_{n+1}} = \bar{\partial}^{-1}u_{\tau_n} + 2iu\bar{\partial}^{-1}(v\bar{\partial}^{-1}u_{\tau_n} - u\bar{\partial}^{-1}v_{\tau_n}), \quad (7a)$$

$$v_{\tau_{n+1}} = -\bar{\partial}^{-1}v_{\tau_n} + 2iv\bar{\partial}^{-1}(u\bar{\partial}^{-1}v_{\tau_n} - v\bar{\partial}^{-1}u_{\tau_n}), \quad (7b)$$

- **The KP hierarchy (Sato theory):**

$$w_{t_n} = K_n, \quad n \in \mathbb{Z}^+. \quad (8)$$

MAIN RESULTS

Starting from (3) and the initial value condition:

$$\mathbf{V}_{x_{n+1}} \mathbf{V}^{-1} = -\mathbf{U}_{x_n}, \quad \mathbf{V}_{x_0} = [\mathbf{V}, \mathbf{a}], \quad \mathbf{U}_{x_0} = [\mathbf{U}, \mathbf{a}]. \quad (9)$$

as well as the following expansion:

$$\mathbf{V} := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \mathbf{U} := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & -U_{11} \end{pmatrix}, \quad V_{11}V_{22} - V_{12}V_{21} = 1,$$

several integrable hierarchies can be constructed from (9):

- **To the AKNS hierarchy:** Following definitions solve (6):

$$q(x, t_n, t_{n+1}) := U_{21}(x_1, x_n, x_{n+1}), \quad (10a)$$

$$r(x, t_n, t_{n+1}) := U_{12}(x_1, x_n, x_{n+1}). \quad (10b)$$

- **To the KN hierarchy:** Following definitions solve (7):

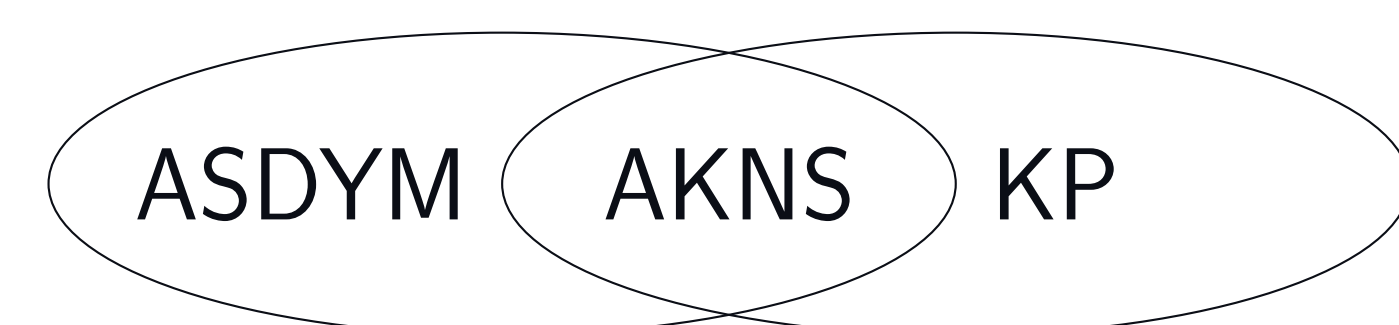
$$u(\xi, \tau_{n-1}, \tau_n) := \partial_{x_{-1}} U_{21}(x_{-1}, x_{-n+1}, x_{-n}), \quad (11a)$$

$$v(\xi, \tau_{n-1}, \tau_n) := i\partial_{x_{-1}} \left(\frac{V_{12}(x_{-1}, x_{-n+1}, x_{-n})}{V_{22}(x_{-1}, x_{-n+1}, x_{-n})} \right). \quad (11b)$$

- **To the KP hierarchy:** Following definition solves (8):

$$w(x, y, t_n) := \partial_{x_1}^{-1} U_{11}(x_1, x_2, x_n). \quad (12)$$

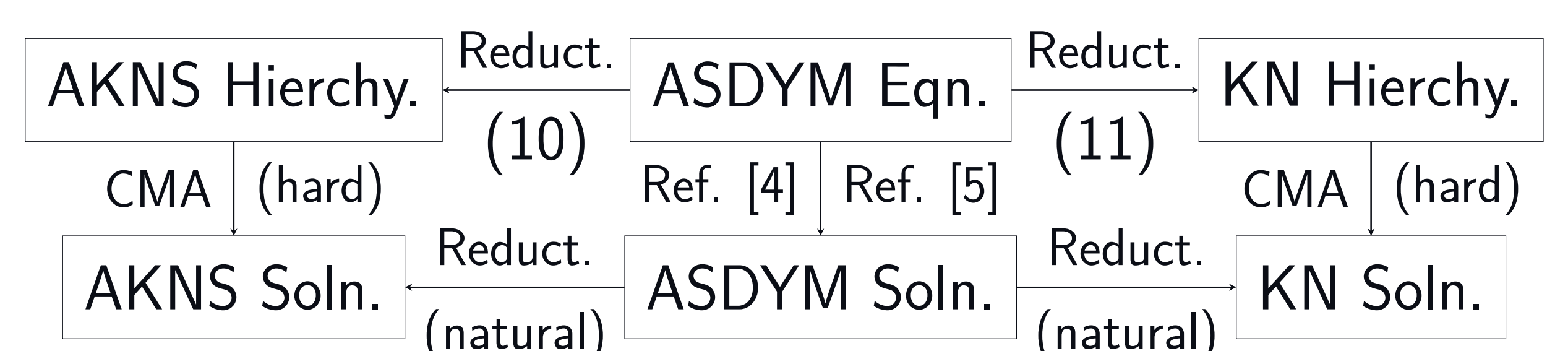
One can verify the relation $w_x = qr$, which is the square eigenfunction symmetry of the KP hierarchy, leads to the AKNS hierarchy. Their relationship can be illustrated by a Venn diagram:



Ward's Conjecture of the KP equation

It has been proved that ASDYM can be reduced to the KP equation [2], whereas the GPFs are of operator-valued. This is usually not be considered as an example of Ward's conjecture. However, our construction provides with a set of GPFs of the KP equation without using any operators:

$$A_z = \begin{pmatrix} qr & r_x \\ q_x - qr \end{pmatrix}, \quad A_w = \begin{pmatrix} q_x r - q r_x & q_{xx} + 2q^2 r \\ -r_{xx} - 2qr^2 & q r_x - q_x r \end{pmatrix}, \quad A_{\bar{z}} = A_{\bar{w}} = 0,$$



A brief framework of the present paper