

On soliton preserving anti-self-dual Yang-Mills reductions

Shangshuai Li (adjoint work with **Ken-ichi Maruno** and **Da-jun Zhang**)

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Mini workshop on ASD Yang-Mills Equations

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- Isotropic Heisenberg model $\rightarrow$ Wadati-Konno-Ichikawa (WKI) equation

Our Goal: By performing dimensional reduction on ASDYM matrices, the AKNS system, the Miura link between NLS and GI, the gauge transformation between DNLS equations, the isotropic Heisenberg model can be realized directly.

The Anti-Self-Dual Yang-Mills Matrix Formulations

Preliminaries of the Anti-Self-Dual Yang-Mills Theory

Let G be the gauge group and $\mathfrak{g}$ be its associate Lie algebra. Let $(z, \tilde{z}, w, \tilde{w})$ be complexified space-time coordinates and $A_z, A_w \in \mathfrak{g}$ be gauge potentials, λ be the spectral parameter, the ASDYM equation read as:

$$[\partial_z - \lambda \partial_{\tilde{w}} + A_z, \partial_w - \lambda \partial_{\tilde{z}} + A_w] = 0,$$

where the gauge potentials read as:

$$A_z = \partial_{\tilde{w}} K = -(\partial_z J) J^{-1}, \quad A_w = \partial_{\tilde{z}} K = -(\partial_w J) J^{-1}.$$

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The compatible condition leads to two equations:

$$\partial_{\tilde{z}}((\partial_z J) J^{-1}) - \partial_{\tilde{w}}((\partial_w J) J^{-1}) = 0, \quad (\text{J-matrix formulation})$$

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K - [\partial_{\tilde{z}} K, \partial_{\tilde{w}} K] = 0. \quad (\text{K-matrix formulation})$$

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Let $P = J^{-1}$ be the inverse of J -matrix, it yields another form of the J -matrix equation:

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Similarly, we can introduce the Q -matrix by

$$\partial_w Q = -(\partial_{\bar{z}}P)P^{-1}, \quad \partial_z Q = -(\partial_{\bar{w}}P)P^{-1}, \quad (\text{Miura link between P and Q})$$

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Remark 1: Usually we only consider the J -matrix and K -matrix, it is enough in most of the research. But in this talk, I will show the P -matrix and Q -matrix is also very important in our reduction framework.

The ASDYM Reduction Framework

Dimensional Reduction Constraint

To reduce the ASDYM equations to $(2 + 1)$ -dimensional integrable systems, we assume $G = SL(2, \mathbb{C})$ and eliminate the $\tilde{w}$ -derivative of J via:

$$\partial_{\tilde{w}} J = [J, E], \quad E = \frac{1}{2} \sigma_3, \quad \sigma_3 = \text{Diag}(1, -1). \quad (\text{Dimensional reduction condition})$$

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Lemma 1: Suppose the above constraint holds, then we have

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Remark 2: Suppose $A \in \{J, K, P, Q\}$, then we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \partial_{\tilde{w}} A = [A, E] = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}.$$

This indicates A admits the decomposition

$$A(z, \tilde{z}, w, \tilde{w}) = \exp(-E\tilde{w}) A'(z, \tilde{z}, w) \exp(E\tilde{w}).$$

Reduction to the NLS Equation

The dimensional constraint $\partial_{\tilde{w}}K = [K, E]$ reduces the K-matrix formulation:

$$\begin{aligned}K_{11} &= \partial_z^{-1}(K_{12}K_{21}), \quad K_{22} = -\partial_z^{-1}(K_{21}K_{12}), \\ \partial_z\partial_{\bar{z}}K_{12} &= -\partial_wK_{12} - (\partial_{\bar{z}}K_{11})K_{12} + K_{12}(\partial_{\bar{z}}K_{22}), \\ \partial_z\partial_{\bar{z}}K_{21} &= \partial_wK_{21} + (\partial_{\bar{z}}K_{22})K_{21} - K_{21}(\partial_{\bar{z}}K_{11}).\end{aligned}$$

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Redefining $(r, q) := (K_{21}, -K_{12})$, we obtain the $(2+1)$ -d NLS system:

$$r_w = r_{z\tilde{z}} - 2r\partial_z^{-1}(qr)_{\tilde{z}}, \tag{NLS-1}$$

$$q_w = -q_{z\tilde{z}} + 2q\partial_z^{-1}(rq)_{\tilde{z}}. \tag{NLS-2}$$

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Through coordinate transformation $(x, y, t) := (iz, i\tilde{z}, iw)$ and conjugate condition $q = r^*$, this yields the $(2+1)$ -d NLS equation

$$iq_t = q_{xy} + 2q\partial_x^{-1}(|q|^2)_y.$$

Reduction to the NLS Equation

$$\text{K-matrix formulation} \xrightarrow[\substack{\partial_{\tilde{w}} K = [K, E] \\ (r, q) := (K_{21}, -K_{12})}]{(2+1)\text{-d NLS system}}$$

Figure 1: From 4-d ASDYM to (2+1)-d NLS

Reduction to the GI Equation

Reduction to (2+1)-d GI system is based on the result of (2+1)-d NLS system. By expanding $(\partial_z J)J^{-1} = \partial_{\tilde{w}} K = [K, E]$, it indicates

$$\begin{pmatrix} \partial_z J_{11} & \partial_z J_{12} \\ \partial_z J_{21} & \partial_z J_{22} \end{pmatrix} = \begin{pmatrix} K_{12} J_{21} & K_{12} J_{22} \\ -K_{21} J_{11} & -K_{21} J_{12} \end{pmatrix}.$$

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Let $(p, q, r) = (J_{21} J_{11}^{-1}, -K_{12}, K_{21})$, they yield the relation:

$$p_z = -r + p^2 q, \tag{MT}$$

which is the Miura transformation between the NLS and GI equations
[\[Takei-Kikuchi-2004\]](#).

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which is the Miura transformation between the NLS and GI equations [\[Takei-Kikuchi-2004\]](#). Thus (p, q) yields the (2+1)-d GI system

$$p_w = p_{z\bar{z}} - 2p\partial_z^{-1}(pq_z + p^2 q^2)_{\bar{z}}, \quad (\text{GI-1})$$

$$q_w = -q_{z\bar{z}} - 2q\partial_z^{-1}(p_z q - p^2 q^2)_{\bar{z}}. \quad (\text{GI-2})$$

Reduction to the GI Equation

Proof: From the second equation of NLS system

$$q_w = -q_{z\bar{z}} + 2q\partial_z^{-1}(rq)_{\bar{z}}, \quad (\text{NLS-2})$$

we substitute $r = -p_z + p^2q$ into it, which gives rise to

$$q_w = -q_{z\bar{z}} - 2q\partial_z^{-1}(p_zq)_{\bar{z}} + 2q\partial_z^{-1}(p^2q^2)_{\bar{z}}. \quad (\text{GI-2})$$

On the other hand, the w -derivative of (MT) shows

$$r_w = (2pq - \partial_z)p_w + p^2q_w,$$

which along with (NLS-1) and (NLS-2) indicates an equation of:

$$p_w = p_{z\bar{z}} - 2p\partial_z^{-1}(pq_z)_{\bar{z}} - 2p\partial_z^{-1}(p^2q^2)_{\bar{z}}. \quad (\text{GI-1})$$

Reduction to the GI Equation

Through coordinate transformation $(x, y, t) := (iz, i\tilde{z}, iw)$ and conjugate condition $q = p^*$, this yields the (2+1)-d GI equation:

$$iq_t = q_{xy} - 2i\partial_x^{-1}(q^2 q_x^*)_y + 2q\partial_x(|q|^4)_y.$$

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We summarize this part with the following figure:

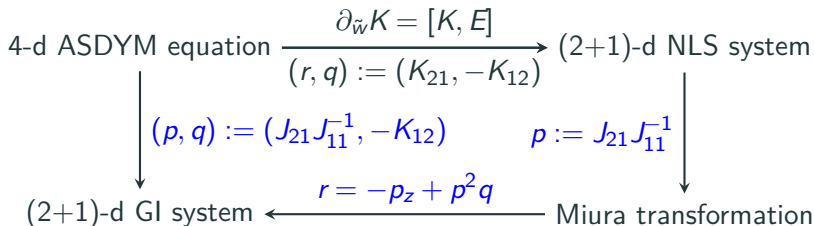


Figure 2: From 4-d ASDYM to (2+1)-d GI

Gauge Transformation and the GDNLS System

Other than GI equation, the DNLS equations are known to have three types, the **Kaup-Newell (KN)** type and the **Chen-Lee-Liu (CLL)** type. They are linked via gauge transformation [[Wadati-Sogo-1983](#)]. Moreover, the three equations can be viewed in one generalized DNLS equation.

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For GI system, the gauge factor is introduced as

$$s = \exp(-\partial_z^{-1}(pq)) = \exp\left(\int \frac{K_{12}J_{21}}{J_{11}}dz\right) = \exp\left(\int \frac{J_{11,z}}{J_{11}}dz\right) = J_{11},$$

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Then we can apply the gauge transformation to define

$$(u, v) := (ps^\gamma, s^{-\gamma}q) = (J_{21}J_{11}^{\gamma-1}, -J_{11}^{-\gamma}K_{12}).$$

Gauge Transformation and the GDNLS System

Then (u, v) yields the (2+1)-d GDNLS system

$$\begin{aligned} u_w = & u_{z\bar{z}} + \gamma u \partial_z^{-1} (u_z v)_{\bar{z}} + \gamma u_z \partial_z^{-1} (uv)_{\bar{z}} \\ & + 2(\gamma - 1) u \partial_z^{-1} (uv_z)_{\bar{z}} - (\gamma - 1)(\gamma - 2) u \partial_z^{-1} (u^2 v^2)_{\bar{z}}, \end{aligned} \quad (\text{GDNLS-1})$$

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Through coordinate transformation $(x, y, t) := (iz, i\bar{z}, iw)$ and conjugate condition $q = p^*$, $s^* = s^{-1}$, this yields the (2+1)-d GDNLS equation:

$$\begin{aligned} iu_t + u_{xy} - i\gamma u \partial_x^{-1} (u_x u^*)_y - i\gamma u_x \partial_x^{-1} (|u|^2)_y \\ + 2i(\gamma - 1) u \partial_x^{-1} (uu_x^*)_y + (\gamma - 1)(\gamma - 2) u \partial_x^{-1} (|u|^4)_y = 0. \end{aligned} \quad (\text{GDNLS})$$

Gauge Transformation and the GDNLS System

Realizations of three types of DNLS equations in ASDYM reduction:

- (2+1)-d GI system: $(p, q) = (J_{21}J_{11}^{-1}, -K_{12})$
- (2+1)-d CLL system: $(\tilde{p}, \tilde{q}) = (ps, qs^{-1}) = (J_{21}, -J_{11}^{-1}K_{12})$
- (2+1)-d KN system: $(\hat{p}, \hat{q}) = (ps^2, qs^{-2}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12})$

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- (2+1)-d KN system: $(\hat{p}, \hat{q}) = (ps^2, qs^{-2}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12})$

Let $\gamma = 1$, defining $(\tilde{p}, \tilde{q}) = (ps, qs^{-1})$, it yields (2+1)-d CLL:

$$\tilde{p}_w = \tilde{p}_{z\bar{z}} + \tilde{p}\partial_z^{-1}(\tilde{p}_z\tilde{q})_{\bar{z}} + \tilde{p}_z\partial_z^{-1}(\tilde{p}\tilde{q})_{\bar{z}}, \quad (\text{CLL-1})$$

$$\tilde{q}_w = -\tilde{q}_{z\bar{z}} + \tilde{q}\partial_z^{-1}(\tilde{p}\tilde{q}_z)_{\bar{z}} + \tilde{q}_z\partial_z^{-1}(\tilde{p}\tilde{q})_{\bar{z}}. \quad (\text{CLL-2})$$

Let $\gamma = 2$, defining $(\hat{p}, \hat{q}) = (ps^2, qs^{-2})$, it yields (2+1)-d KN:

$$\hat{p}_w = \hat{p}_{z\bar{z}} + 2(\hat{p}\partial_z^{-1}(\hat{p}\hat{q})_{\bar{z}})_z, \quad (\text{KN-1})$$

$$\hat{q}_w = -\hat{q}_{z\bar{z}} + 2(\hat{q}\partial_z^{-1}(\hat{p}\hat{q})_{\bar{z}})_z. \quad (\text{KN-2})$$

Gauge Transformation and the GDNLS System

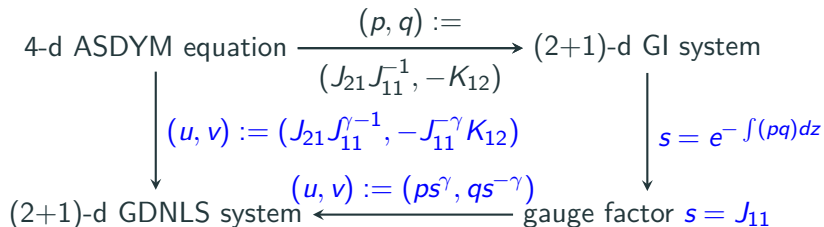


Figure 3: From 4-d ASDYM to (2+1)-d GDNLS

The Isotropic Heisenberg Model and WKI Equation

The isotropic Heisenberg equation is geometrically equivalent to the NLS equation [\[Zakharov-Takhtadzhyan-1979\]](#). For the (2+1)-d focusing NLS equation:

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After a very complicated transformation (cf. [Wadati-Sogo-1983]), it gives rise to the isotropic Heisenberg equation equation of Myrzakulov-I type:

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x, \quad (\text{M-I-1})$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y). \quad (\text{M-I-2})$$

where $\mathbf{S} = (s_1, s_2, s_3) \in \mathbb{R}^3$ be the coordinates of a point on the unit sphere in $\mathbb{R}^3$.

The Isotropic Heisenberg Model and WKI Equation

We will show this isotropic Heisenberg model can be directly derived from the ASDYM matrix formulations.

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$$S := J^{-1} \sigma_3 J,$$

yielding $S^2 = I$, $|S| = -1$, and $Tr(S) = 0$.

The Isotropic Heisenberg Model and WKI Equation

We will show this isotropic Heisenberg model can be directly derived from the ASDYM matrix formulations. Let us define the S -matrix by

$$S := J^{-1} \sigma_3 J,$$

yielding $S^2 = I$, $|S| = -1$, and $Tr(S) = 0$. The dimensional reduction constraint $\partial_{\tilde{w}} J = [J, E]$ indicates that one can rewrite $\partial_z Q$ in terms of S as

$$\partial_z Q = J^{-1} [J, E] = E - J^{-1} E J = E - \frac{1}{2} S. \quad (\text{Q-S-system-1})$$

Similarly, the relation $\partial_w Q = J^{-1} (\partial_{\tilde{z}} J)$ indicates

$$\partial_{\tilde{z}} S = -J^{-1} (\partial_{\tilde{z}} J) J^{-1} \sigma_3 J + J^{-1} \sigma_3 (\partial_{\tilde{z}} J) = [S, \partial_w Q]. \quad (\text{Q-S-system-2})$$

The Isotropic Heisenberg Model and WKI Equation

Then the S -matrix yields the following (2+1)-d isotropic Heisenberg model:

$$\partial_w S + 2\kappa(\partial_z S) + \frac{1}{2}[S, \partial_z \partial_{\bar{z}} S] = 0, \quad (\text{IHM-1})$$

$$\partial_z \kappa = \frac{1}{8} \text{Tr}([\partial_z S, \partial_{\bar{z}} S] S). \quad (\text{IHM-2})$$

Proof: By a complicated calculation and use some properties of $\mathfrak{sl}(2)$ Lie algebra.

(Sorry that we have to skip this part due to the limit time.....)

The Isotropic Heisenberg Model and WKI Equation

Then the S -matrix yields the following (2+1)-d isotropic Heisenberg model:

$$\partial_w S + 2\kappa(\partial_z S) + \frac{1}{2}[S, \partial_z \partial_{\bar{z}} S] = 0, \quad (\text{IHM-1})$$

$$\partial_z \kappa = \frac{1}{8} \text{Tr}([\partial_z S, \partial_{\bar{z}} S] S). \quad (\text{IHM-2})$$

Proof: By a complicated calculation and use some properties of $\mathfrak{sl}(2)$ Lie algebra.

(Sorry that we have to skip this part due to the limit time.....)

Through the coordinate transformation $(x, y, t) := (iz, i\bar{z}, iw) \in \mathbb{R}^3$ and the conjugate condition $S = S^\dagger$, we derive the isotropic Heisenberg equation. Suppose the S -matrix is decomposed to

$$S = \mathbf{S} \cdot \boldsymbol{\sigma} = s_1 \sigma_1 + s_2 \sigma_2 + s_3 \sigma_3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices.

The Isotropic Heisenberg Model and WKI Equation

This decomposition implies

$$S = \begin{pmatrix} s_3 & s_1 - is_2 \\ s_1 + is_2 & -s_3 \end{pmatrix}.$$

Let $u = -2\kappa$, we rewrite the equation of S -matrix as

$$iS_t - iuS_x - \frac{1}{2}[S, S_{xy}] = 0,$$

$$iu_x = -\frac{1}{4}\text{Tr}([S_x, S_y]S).$$

This system is equivalent to the M-I equation:

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x,$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y).$$

The Isotropic Heisenberg Model and WKI Equation

Consider the (1+1)-d reduced form of the M-I equation by imposing the constraint $\partial_x = \partial_y$ and assuming $u = 0$, it becomes (cf. (4.3) in [\[Wadati-Sogo-1983\]](#))

$$iS_t = \frac{1}{2}[S, S_{xx}].$$

By introducing the coordinate transformation and new definition (cf. (4.8) and (4.9) in [\[Wadati-Sogo-1983\]](#))

$$X := - \int f dx, \quad T := t, \quad \hat{q} := -i \frac{g}{f} = -i \frac{s_1 - is_2}{s_3},$$

it yields the WKI equation:

$$\hat{q}_T \pm \left(\frac{\hat{q}}{\sqrt{1 + |\hat{q}|^2}} \right)_{xx} = 0. \quad (\text{WKI})$$

Some Applications

On Deriving the Bilinear Transformation

The bilinear transformation for ASDYM equations has been given by [\[Ohta-2024\]](#):

$$J := \frac{G}{f}, \quad J^{-1} := \frac{S}{f}, \quad K := \frac{H}{f},$$

which transforms the gauge potentials into the following bilinear equations

$$D_z G \cdot S = -2D_{\tilde{w}} H \cdot (fI), \quad D_2 G \cdot S = -2D_{\tilde{z}} H \cdot (fI),$$

where $GS = f^2 I$ and I is the identity matrix.

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For the $SL(2)$ ASDYM equations, we partition them as $G = \{g_{ij}\}$ and $H = \{h_{ij}\}$ ($1 \leq i, j \leq 2$). Thus the entries in J and K can be rewritten as

$$J_{ij} = \frac{g_{ij}}{f}, \quad K_{ij} = \frac{h_{ij}}{f}, \quad (1 \leq i, j \leq 2).$$

On Deriving the Bilinear Transformation

According to the bilinear transformation of the ASDYM reduction framework, we propose the bilinear transformation for the reduced system

- The NLS system: $(r, q) = (K_{21}, -K_{12}) = (h_{21}/f, -h_{12}/f)$.

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- The GI system: $(p, q) = (J_{21}J_{11}^{-1}, -K_{12}) = (g_{21}/g_{11}, -h_{12}/f)$.
- CLL (cf. [\[Nakamura-Chen-1980\]](#)): $(\tilde{p}, \tilde{q}) = (J_{21}, -J_{11}^{-1}K_{12}) = (g_{21}/f, -h_{12}/g_{11})$.
- The KN system: $(\hat{p}, \hat{q}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12}) = (g_{11}g_{21}/f^2, -fh_{12}/g_{11}^2)$.

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- The KN system: $(\hat{p}, \hat{q}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12}) = (g_{11}g_{21}/f^2, -fh_{12}/g_{11}^2)$.
- The gauge factor: $s = J_{11} = g_{11}/f$.
- The GDNLS system (cf. [\[Kakei-Sasa-Satsuma-1998\]](#)):

$$(u, v) = (J_{21}J_{11}^{\gamma-1}, -J_{11}^{-\gamma}K_{12}) = \left(\frac{g_{11}^{\gamma-1}g_{21}}{f^{\gamma}}, -\frac{f^{\gamma-1}h_{12}}{g_{11}^{\gamma}} \right).$$

On Deriving Soliton Solutions

The ASDYM reduction framework starts from the ASDYM matrix formulations, and they have soliton solutions. This indicates if we can apply the dimensional reduction condition to these solitons, then we can have soliton solutions of the reduced systems.

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Question 1: How to solve the ASDYM equations (difficult or simple?)

$$\begin{aligned}\partial_{\tilde{w}}K &= -(\partial_z J)J^{-1}, & \partial_{\tilde{z}}K &= -(\partial_w J)J^{-1}, \\ \partial_w Q &= -(\partial_{\tilde{z}}P)P^{-1}, & \partial_z Q &= -(\partial_{\tilde{w}}P)P^{-1}.\end{aligned}$$

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Question 2: How to achieve the dimensional reduction (are they very strong?)

$$\partial_{\tilde{w}}J = [J, E], \quad \partial_{\tilde{w}}P = [P, E], \quad \partial_{\tilde{w}}K = [K, E], \quad \partial_{\tilde{w}}Q = [Q, E].$$

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We use the Cauchy matrix approach as an example, which generates **bright-type** N -**soliton** solution. For ASDYM equation, its Cauchy matrix structure is given by

The Sylvester equation: $\Xi\Omega - \Omega\Lambda = \eta\theta$.

Dispersion relations:
$$\begin{cases} \partial_z\eta = \Xi(\partial_{\bar{w}}\eta), & \partial_z\theta = (\partial_{\bar{w}}\theta)\Lambda, \\ \partial_w\eta = \Xi(\partial_{\bar{z}}\eta), & \partial_w\theta = (\partial_{\bar{z}}\theta)\Lambda. \end{cases}$$

Master function: $S^{(i,j)} := \theta\Lambda^j\Omega^{-1}\Xi^i\eta$,

where $\Xi, \Lambda \in \mathbb{C}_{N \times N}$ and $\eta \in \mathbb{C}_{N \times 2}$, $\theta \in \mathbb{C}_{2 \times N}$ (cf. [\[Li-Qu-Zhang-2023\]](#)).

Remark 3: If we simply assume Ξ and Λ to be diagonal matrices, then Ω would be dressed Cauchy matrix. [\(I will explain this in the next page\)](#)

On Deriving Soliton Solutions

Suppose Ξ and Λ are both diagonal, then the Cauchy matrix structure is satisfied by the following construction [Ohta-2024]:

$$\begin{aligned}\Xi &= \text{Diag}(\xi_1, \dots, \xi_N), & \eta &= (\phi_{is}(\xi_i z + \tilde{w}, \xi_i w + \tilde{z}))_{N \times 2}, \\ \Lambda &= \text{Diag}(\lambda_1, \dots, \lambda_N), & \theta &= (\psi_{sj}(\lambda_j z + \tilde{w}, \lambda_j w + \tilde{z}))_{2 \times N}, \\ \Omega &= (\Omega_{ij})_{N \times N}, & \Omega_{ij} &= \frac{\sum_{s=1}^2 \phi_{is} \psi_{sj}}{\xi_i - \lambda_j},\end{aligned}$$

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where ϕ_{is} and ψ_{sj} are **plane wave factors**, can be arbitrary functions. If we use the plane wave factors in [Hamanaka-Huang-2022], it gives rise to soliton solution:

$$\begin{aligned}\phi_{is} &= A_{is} \exp(\alpha_{is}(\xi_i z + \tilde{w}) + \beta_{is}(\xi_i w + \tilde{z})), \\ \psi_{sj} &= B_{sj} \exp(\gamma_{sj}(\lambda_j z + \tilde{w}) + \delta_{sj}(\lambda_j w + \tilde{z})).\end{aligned}$$

On Deriving Soliton Solutions

Under this Cauchy matrix structure, we find that $S^{(i,j)}$ will satisfy

$$\begin{aligned} S^{(i+1,j)} - S^{(i,j+1)} &= S^{(0,j)} S^{(i,0)}, \\ \partial_w S^{(i-1,j)} - \partial_{\tilde{z}} S^{(i,j)} + \partial_{\tilde{z}} S^{(0,j)} S^{(i-1,0)} &= 0, \\ \partial_z S^{(i-1,j)} - \partial_{\tilde{w}} S^{(i,j)} + \partial_{\tilde{w}} S^{(0,j)} S^{(i-1,0)} &= 0. \end{aligned}$$

Remark 4: Recently, these relations are found to be identified with the affine coordinates of Sato Grassmann manifold [\[Takasaki-2025\]](#).

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Then the N -soliton solutions of each ASDYM matrix are respectively given by

$$\begin{aligned} J &= I - S^{(-1,0)} = I - \theta \Omega^{-1} \Xi^{-1} \eta, & K &= S^{(0,0)} = \theta \Omega^{-1} \eta, \\ P &= I + S^{(0,-1)} = I + \theta \Lambda^{-1} \Omega^{-1} \eta, & Q &= S^{(-1,-1)} = \theta \Lambda^{-1} \Omega^{-1} \Xi^{-1} \eta. \end{aligned}$$

On Deriving Soliton Solutions

The dimensional reduction condition can be achieved by imposing

$$\partial_{\tilde{w}}\eta = \eta E, \quad \partial_{\tilde{w}}\theta = -E\theta.$$

Then we have $\partial_{\tilde{w}}\Omega = 0$ and

$$\partial_{\tilde{w}}S^{(i,j)} = -E\theta\Lambda^j\Omega^{-1}\Xi^i\eta + \theta\Lambda^j\Omega^{-1}\Xi^i\eta E = [S^{(i,j)}, E].$$

Remark 5: This constraint indicates the plane wave factors has the form of

$$\begin{aligned} \phi_{i1} &= A_{is} \exp\left(\frac{1}{2}(\xi_i z + \tilde{w}) + \beta_{is}(\xi_i w + \tilde{z})\right), \quad \phi_{i2} = A_{is} \exp\left(-\frac{1}{2}(\xi_i z + \tilde{w}) + \beta_{is}(\xi_i w + \tilde{z})\right), \\ \psi_{1j} &= B_{sj} \exp\left(-\frac{1}{2}(\lambda_j z + \tilde{w}) + \delta_{sj}(\lambda_j w + \tilde{z})\right), \quad \psi_{2j} = B_{sj} \exp\left(\frac{1}{2}(\lambda_j z + \tilde{w}) + \delta_{sj}(\lambda_j w + \tilde{z})\right). \end{aligned}$$

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In this talk, we review the relations between some lower-dimensional integrable systems, most of them are related through **Miura transformation** and **gauge transformation**.

One of the reason that we can derive them systems directly from the ASDYM equations is because ASDYM equations is characterized by a Miura link (between J and K), and it is a gauge-invariant system.

Usually these integrable systems are described by **the compatibility of Lax pairs**, the formula is very long, and it is a very complicate calculation by hand. But we skip this part to derive these systems directly.

Conclusion

In this method, we use only the ASDYM matrix formulations, and impose a very simple (or weak?) dimensional reduction condition, then many fruitful integrable systems come out.

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In this method, we use only the ASDYM matrix formulations, and impose a very simple (or weak?) dimensional reduction condition, then many fruitful integrable systems come out.

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This method is simple but potential, other than above applications, we can use it to

- Derive multi-component system: By considering the $SL(m_1 + m_2)$ group
- Derive integrable hierarchy: By considering the higher flows of ASDYM equations
- Derive Lax pair of the reduced system: From the Lax pair of ASDYM equations

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Remark 5: I have no time to explain all details here, if anyone is interested in it, you can read my latest submission in "<https://arxiv.org/pdf/2603.23060>".

Thank You for Listening!