

Sato-theoretic Construction of the
anti-self-dual Yang-Mills hierarchy
and the Ward Conjecture

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Integrability, Self-Duality, and Twistor Theory

Outline

- Motivation
- Sato–theoretic construction
- Integrable reductions
- Concluding remarks

Motivation

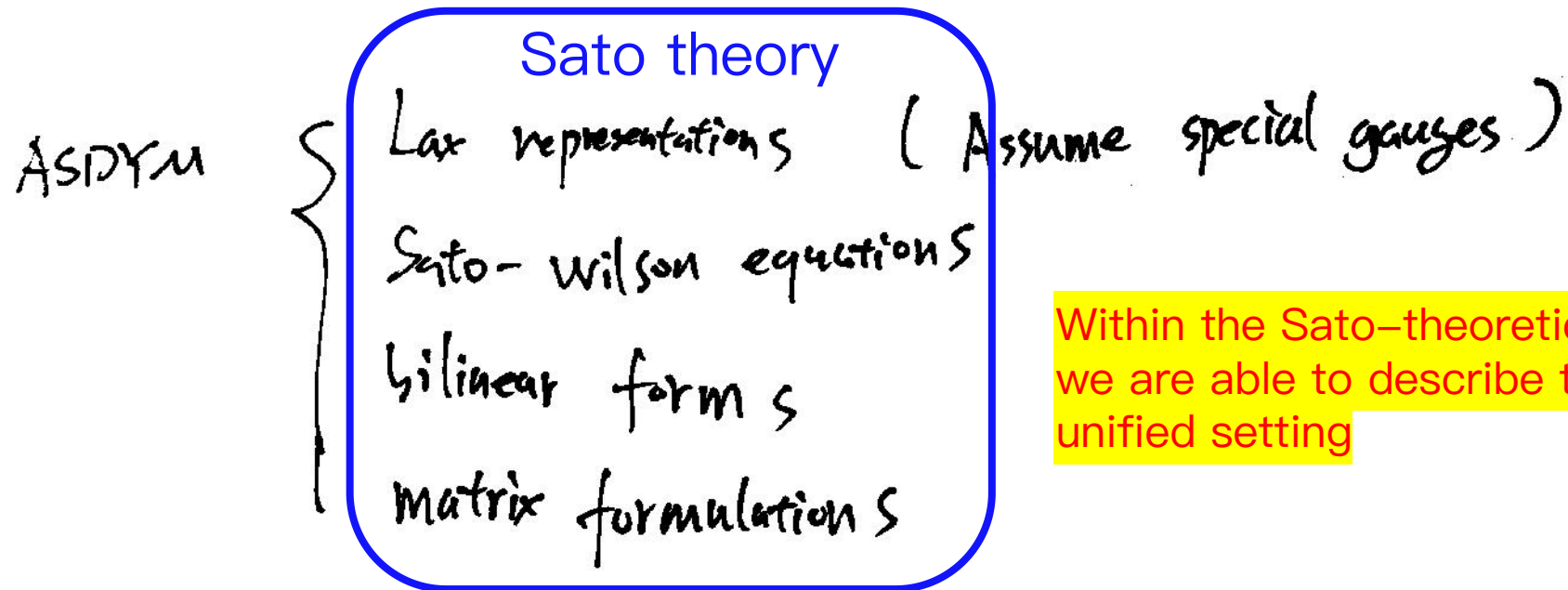
- [Ward conjecture, [Ward1985](#)]
- the anti-self-dual Yang-Mills (ASDYM) equations (or its generalizations) admit reductions to lower-dimensional integrable systems. Commonly known as the Ward conjecture
- e.g. NLS, KdV, KP, sine-Gordon, derivative NLS, AKNS, Painlevé,

ASDYM { Lax representations (Assume special gauges) [[ACH2003](#)]
Sato-Wilson equations [[KIT2001](#)]
bilinear forms [[SOM1998](#)]
matrix formulations [[LMZ2026](#)]
twistor representations [[AW1977](#), [MS1989](#), [MW1996](#)]

Different methods take rather different forms and appear unrelated at first sight.

Motivation

- [Sato theory, [Takasaki1984](#), [KIT2002](#)]
- A Sato–theoretic construction of the ASDYM hierarchy provides a unified framework for describing the Sato–Wilson equation, the auxiliary linear system, the tau function, the ASDYM matrix formulations.
- This could be a unified origin for us to investigate the Ward conjecture.



Within the Sato–theoretic framework,
we are able to describe them in a
unified setting

Motivation

- [Cauchy matrix approach, [LQYZ2022](#), [LQZ2023](#), [LZ2025](#)]
- Recently, a ASDYM reduction framework based on Cauchy matrix approach was developed. It turns out that several classical integrable systems can be derived. Their nonlinear variables are represented in terms of ASDYM matrix entries.

SL(2) gauge group

$$\partial_n K = -(\partial_{n+1} J) J^{-1}, \quad \partial_n := \frac{\partial}{\partial t_n}, \quad n \geq 0,$$

$$\partial_0 K = \left[K, \frac{1}{2} \sigma_3 \right], \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

INPUT

[LMZ2026]

<https://arxiv.org/abs/2603.23060>

- The AKNS hierarchy (2.13): $(r, q) = (K_{21}, -K_{12})$.
- The GI hierarchy (2.18): $(p, q) = (J_{21} J_{11}^{-1}, -K_{12})$.
- The gauge factor between DNLS hierarchies (2.25): $s = J_{11}$.
- The GDNLS hierarchy (2.28): $(u, v) = (J_{21} J_{11}^{\gamma-1}, -J_{11}^{-\gamma} K_{12})$.
- The KP hierarchy (2.41): $u = K_{11,x}$.
- The mKP hierarchy (2.50): $v = -(\ln J_{11})_x$.

OUTPUT

Motivation

Cauchy matrix approach
and

Sato Grassmannian

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- [Cauchy matrix approach and Sato Grassmannian, [Takasaki2025](#)]
- More Recently, the relationship between Cauchy matrix approach and Sato theory was revealed. The master functions satisfy the same quadratic equations of affine coordinates.
- It builds a bridge between the Sato theory and the Cauchy matrix approach.

Properties of w_{ij} 's

- $W(\zeta)$ and $W(\zeta)^{-1}$ can be recovered from w_{ij} 's as

$$W(\zeta) = I - \sum_{j=0}^{\infty} w_{0j} \zeta^{-j-1}, \quad W(\zeta)^{-1} = I + \sum_{i=0}^{\infty} w_{i0} \zeta^{-i-1}.$$

- w_{ij} 's satisfy (and are characterized by) the equations

$$\begin{aligned} w_{i+1,j} &= w_{i,j+1} + w_{i0} w_{0j}, \\ \frac{\partial w_{ij}}{\partial y_k} &= \frac{\partial w_{i+1,j}}{\partial y_{k-1}} - w_{i0} \frac{\partial w_{0j}}{\partial y_{k-1}}, \\ \frac{\partial w_{ij}}{\partial z_k} &= \frac{\partial w_{i+1,j}}{\partial z_{k-1}} - w_{i0} \frac{\partial w_{0j}}{\partial z_{k-1}} \end{aligned}$$

Properties of $S^{(i,j)}$

$S^{(i,j)}$'s satisfy the algebraic and differential equations

$$\begin{aligned} S^{(i+1,j)} &= S^{(i,j+1)} + S^{(i,0)} S^{(0,j)}, \\ \frac{\partial S^{(i,j)}}{\partial y_k} &= \frac{\partial S^{(i+1,j)}}{\partial y_{k-1}} - S^{(i,0)} \frac{\partial S^{(0,j)}}{\partial y_{k-1}}, \\ \frac{\partial S^{(i,j)}}{\partial z_k} &= \frac{\partial S^{(i+1,j)}}{\partial z_{k-1}} - S^{(i,0)} \frac{\partial S^{(0,j)}}{\partial z_{k-1}}, \end{aligned}$$

which are identical to the equations for w_{ij} 's. Thus $S^{(i,j)}$ for $i, j = 0, 1, \dots$, can be identified with the affine coordinates in the Grassmannian description of the ASDYM hierarchy.

Motivation

- Based on these known results, we aim to develop the Sato–theorectic construction of the ASDYM hierarchy, and use it to investigate the Ward conjecture.
- **The main results:**
 - 1. We extend the affine coordinates $(w_{ij})_{i,j \geq 0}$ on the big cell of the Sato Grassmannian to a bi-infinite matrix array $(w_{ij})_{i,j \in \mathbb{Z}}$.

$$\xi = \begin{pmatrix} \delta_{ij} I \\ w_{i,-j-1} \end{pmatrix}_{i \in \mathbb{Z}, j < 0} = \begin{pmatrix} \ddots & \vdots & \vdots \\ \cdots & I & 0 \\ \cdots & 0 & I \\ \cdots & w_{01} & w_{00} \\ \cdots & w_{11} & w_{10} \\ \ddots & \vdots & \vdots \end{pmatrix} \xrightarrow{\quad} \mathbf{w} := \begin{pmatrix} & \vdots & \vdots & \vdots \\ \cdots & w_{-1,-1} & w_{-1,0} & w_{-1,1} & \cdots \\ \cdots & w_{0,-1} & w_{00} & w_{01} & \cdots \\ \cdots & w_{1,-1} & w_{10} & w_{11} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

Motivation

- 2. We revisit the continuous Sato–Wilson equations, auxiliary linear system, and coordinate flows. We use **Miwa shifts** to derive their discrete analogues, thereby placing continuous and discrete reductions of the ASDYM hierarchy within the same framework.
- 3. The Sato–theoretic origin of the **dimensional reduction constraint** employed in [LMZ2026] is identified. This reduction recovers several classical integrable systems, with their nonlinear variables expressible in terms of specific relative coordinates.

$$\mathbb{T}_h := \exp \left(\sum_{n \geq 1} \frac{h^n}{n} \partial_n \right), \quad \mathbb{T}_h X(t) = X(t + [h]), \quad [h] := \left(0, h, \frac{h^2}{2}, \frac{h^3}{3}, \dots \right).$$

$$\partial_0 K = \left[K, \frac{1}{2} \sigma_3 \right], \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Sato-theoretic construction

- [Dressing matrices]

Fix constants $0 < r < R$, and set

$$D_0 := \{\lambda \in \mathbb{C} \mid |\lambda| < R\}, \quad D_\infty := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid |\lambda| > r\}.$$

Let $g(\lambda)$ be holomorphic on $D_0 \cap D_\infty$, and suppose

$$g(\lambda) = W_0^{-1}(\lambda) W_\infty(\lambda)$$

Riemann–Hilbert problem

where W_0 is holomorphic in D_0 , W_∞ is holomorphic in D_∞ and $W_\infty(\infty) = I_N$

Sato–theoretic construction

- [Dressing matrices]
- We assume W_0 and W_∞ expands as (After adding time dependence, we will show that, they are the dressing matrices, and satisfy the Sato–Wilson equation)

$$W_\infty(\lambda) = I_N - \sum_{n \geq 1} w_n \lambda^{-n}, \quad W_0(\lambda) = \hat{w}_0 + \sum_{n \geq 1} \hat{w}_n \lambda^n.$$

- We write them into a unified expression

$$W_a(\lambda) = I_N - \varepsilon_a \sum_{j \in \mathbb{Z}_a} w_{0j} \lambda^{-j-1}, \quad a \in \{0, \infty\}.$$

- where

$$\mathbb{Z}_0 := \mathbb{Z}_{<0}, \quad \mathbb{Z}_\infty := \mathbb{Z}_{\geq 0}, \quad \varepsilon_0 := -1, \text{ and } \varepsilon_\infty := 1.$$

- and

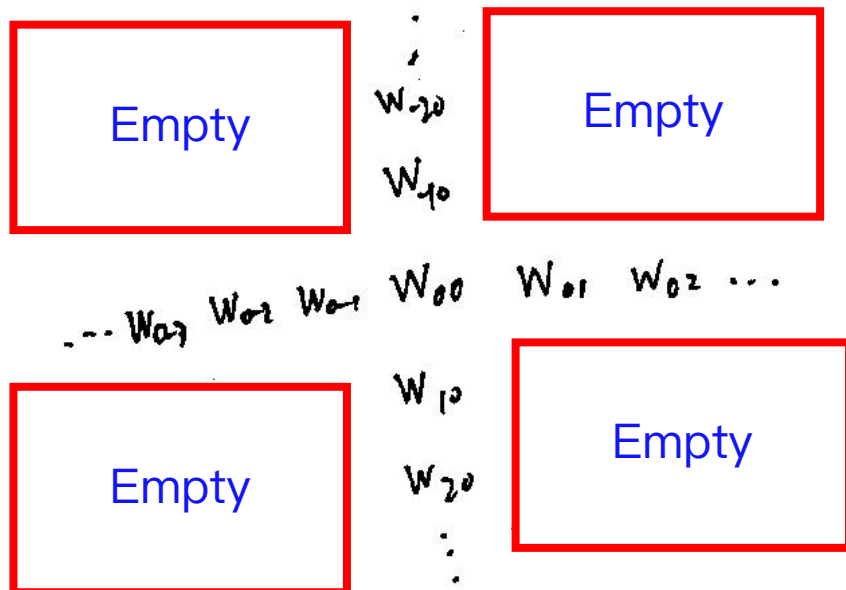
$$w_{0,-1} = \hat{w}_0 - I_N, \quad w_{0,-n-1} = \hat{w}_n, \quad w_{0,n-1} = w_n, \quad n \geq 1.$$

Sato-theoretic construction

- [Dressing matrices]
- Likewise, we write the inverse as

$$W_a(\lambda)^{-1} = I_N + \varepsilon_a \sum_{i \in \mathbb{Z}_a} w_{i0} \lambda^{-i-1}, \quad a \in \{0, \infty\}.$$

- Remark: Equation (1) determines (w_{0j}) , while (2) determines (w_{i0})



We call (w_{0j}) the zeroth column

We call (w_{i0}) the zeroth row

Sato-theoretic construction

- [Generating function, [JM1981](#)]

- To fill up the empty part, we introduce a generating function.

$$\mathcal{W}_{ab}(\eta, \zeta) := \frac{W_a(\eta)^{-1} W_b(\zeta) - I_N}{\zeta - \eta}.$$

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & w_{-1,-1} & w_{-1,0} & w_{-1,1} & \cdots \\ \cdots & w_{0,-1} & w_{00} & w_{01} & \cdots \\ \cdots & w_{1,-1} & w_{10} & w_{11} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

- [Relative coordinates]

- The expansion gives the definition for $w_{\{ij\}}$ with i, j in $\mathbb{Z}^2$, we call them the **relative coordinates**

$$\mathcal{W}_{ab}(\eta, \zeta) = \varepsilon_a \varepsilon_b \sum_{i \in \mathbb{Z}_a} \sum_{j \in \mathbb{Z}_b} w_{ij} \eta^{-i-1} \zeta^{-j-1}.$$

- (only $i, j \geq 0$ part can be identified as affine coordinates)

$$W_{\infty} = \frac{W_{\infty}(\eta)^{-1} W_{\infty}(\zeta) - I_N}{\zeta - \eta}$$

For example, the (infty, infty)–

sector expansion shows:

[[Takasaki2025](#)]

$$W_{\infty} = \sum_{i \geq 0} \sum_{j \geq 0} w_{ij} \eta^{-i-1} \zeta^{-j-1}$$

Sato–theoretic construction

- [Recurrence relation]

Theorem 2.3 (Recurrence relation). *The relative coordinates satisfy*

$$w_{i+1,j} - w_{i,j+1} = w_{i0}w_{0j}, \quad i, j \in \mathbb{Z}.$$

- This recurrence can be regarded as a generating relation from zeroth column and zeroth row to a bi-infinite array.
- Remark: The generating function establish a link between dressing matrices and relative coordinates.

Sato–theoretic construction

- [Sato–Wilson equation]

In this talk, we only consider ASDYM hierarchy with single hierarchy time variables

Let $\mathbf{t} = (t_0, t_1, t_2, \dots)$ be a countable set of hierarchy variables with weights $\text{wt}(t_n) := n + 1$. We allow the dressing matrices to depend on $\mathbf{t}$ and set

$$g(\mathbf{t}, \lambda) := W_0(\mathbf{t}, \lambda)^{-1} W_\infty(\mathbf{t}, \lambda), \quad \lambda \in D_0 \cap D_\infty. \quad (3.1)$$

For $n \geq 1$, introduce $\mathcal{D}_n(\lambda) := \partial_n - \lambda \partial_{n-1}$. The Sato–Wilson evolution is governed by

$$\mathcal{D}_n(\lambda) g(\mathbf{t}, \lambda) = 0, \quad n \geq 1. \quad (3.2)$$

Theorem 3.1 (Sato–Wilson equations). *Assume that g satisfies (3.2). Then, for each $n \geq 1$, there exists a unique λ -independent matrix function $A_n(\mathbf{t})$ such that*

$$\partial_n W_a(\mathbf{t}, \lambda) = \lambda \partial_{n-1} W_a(\mathbf{t}, \lambda) - A_n(\mathbf{t}) W_a(\mathbf{t}, \lambda), \quad a \in \{0, \infty\}. \quad (3.3)$$

- Remark: The generating function establish a link between dressing matrices and relative coordinates. Thus, the dressing matrices satisfy the Sato–Wilson equation, then, the relative coordiantes satisfy corresponding coordinate flow.

Sato-theoretic construction

- [Coordinate flow]

Theorem 3.4 (Coordinate flows). *Under the assumptions of Theorem 3.1, the relative coordinates satisfy*

$$\partial_n w_{ij} - \partial_{n-1} w_{i+1,j} = -w_{i0} \partial_{n-1} w_{0j}, \quad n \geq 1, \quad i, j \in \mathbb{Z}. \quad (3.8)$$

- Let $J := I + w_{\{0,-1\}}$, $K := -w_{\{0,0\}}$, then

$$\partial_n K = -(\partial_{n+1} J) J^{-1}, \quad \partial_n := \frac{\partial}{\partial t_n}, \quad n \geq 0,$$

- Similarly let $P := I - w_{\{-1,0\}}$, $Q := w_{\{-1,-1\}}$, then

$$\partial_n Q = -(\partial_{n-1} P) P^{-1} = J^{-1} \partial_{n-1} J.$$

$(i,j)=(-1,0)$

$(i,j)=(-1,-1)$

The diagram shows an infinite matrix with rows and columns indexed by integers. A red box highlights the 2x2 submatrix at indices $(-1,0)$ and $(0,-1)$, containing the elements $w_{-1,-1}$, $w_{-1,0}$, $w_{0,-1}$, and w_{00} . The matrix is partitioned by dashed blue lines, and a blue arrow points to the right from the matrix.

Sato-theoretic construction

- [Baker function and auxiliary linear system/Lax representation]

Fix a constant matrix $C \in \text{Mat}_N(\mathbb{C})$, and define

$$E(\mathbf{t}, \lambda) := \exp(-C\xi(\mathbf{t}, \lambda)), \quad \xi(\mathbf{t}, \lambda) := \sum_{n \geq 0} t_n \lambda^n. \quad (3.11)$$

For every $n \geq 0$, differentiation gives

$$\partial_n E(\mathbf{t}, \lambda) = -C\lambda^n E(\mathbf{t}, \lambda), \quad (3.12)$$

Hence, $\mathcal{D}_n(\lambda)E = 0$, so that E serves as the vacuum state. Define the Baker functions

$$\Psi_a(\mathbf{t}, \lambda) := W_a(\mathbf{t}, \lambda)E(\mathbf{t}, \lambda), \quad a \in \{0, \infty\}.$$

Baker function can be viewed as
a dressed exponential function

Equations (3.3) and (3.12) give the auxiliary linear system

$$(\partial_n - \lambda \partial_{n-1} + A_n)\Psi_a = 0, \quad n \geq 1, \quad (3.13)$$

and the corresponding transition function is

$$F(\mathbf{t}, \lambda) := \Psi_0(\mathbf{t}, \lambda)^{-1} \Psi_\infty(\mathbf{t}, \lambda) = E(\mathbf{t}, \lambda)^{-1} g(\mathbf{t}, \lambda) E(\mathbf{t}, \lambda). \quad (3.14)$$

Sato-theoretic construction

- [Dimensional reduction constraint]

$$F(t, \lambda) := \Psi_0(t, \lambda)^{-1} \Psi_\infty(t, \lambda) = E(t, \lambda)^{-1} g(t, \lambda) E(t, \lambda).$$

- Usually F depends on t . However, if we assume F is independent of all time variables, it leads to the t_0 -flow constraint for g , i.e. $g_{\{t_0\}} = [g, C]$

Proof: Taking the t_n -derivative of F

$$\partial_n F = \partial_n (E^{-1} g E) = E^{-1} (\partial_n g - \lambda^n [g, C]) E$$

$$\text{Since } \partial_n g - \lambda \partial_{n-1} g = 0 \rightarrow \partial_n g - \lambda^n \partial_0 g = 0$$

$$\text{Thus } \partial_0 g = [g, C]$$

Sato-theoretic construction

- [Dimensional reduction constraint]

$$F(t, \lambda) := \Psi_0(t, \lambda)^{-1} \Psi_\infty(t, \lambda) = E(t, \lambda)^{-1} g(t, \lambda) E(t, \lambda).$$

- Usually F depends on t . However, if we assume F is independent of all time variables, it leads to the t_0 -flow constraint for g , i.e. $g_{\{t_0\}} = [g, C]$

Application:

$$\begin{array}{lcl} \partial_0 g = [g, C] & \xrightarrow{g = W_0^{-1} W_\infty} & \partial_0 W_a = [W_a, C], a \in \{0, \infty\} \\ \partial_0 W_a = [W_a, C] & \xrightarrow{W_{ab} = \frac{W_a^{-1}(y) W_b(z) - 1}{z - y}} & \partial_0 W_{ij} = [W_{ij}, C] \\ \partial_0 W_{ij} = [W_{ij}, C] & \xrightarrow{\begin{array}{l} J = 2 + W_{0,-1}, K = -W_{00} \\ P = 2 - W_{-1,0}, Q = W_{-1,-1} \end{array}} & \partial_0 \chi = [\chi, C], \chi \in \{J, P, Q, K\} \end{array}$$

Sato–theoretic construction

- [Dimensional reduction constraint]

We further emphasize that the four ASDYM matrix variables defined in (3.5) and (3.9) satisfy the same t_0 -flow constraint:

$$\partial_0 J = [J, C], \quad \partial_0 K = [K, C], \quad \partial_0 P = [P, C], \quad \partial_0 Q = [Q, C]. \quad (3.18)$$

- This **dimensional reduction constraint** also appears in [LMZ2026]
- At first, we adopt this choice simply because it arises naturally within **the Cauchy matrix structure of the AKNS hierarchy**, though its deeper origin remained unclear.
- From the perspective of Sato theory, however, this constraint can be understood as a consequence of the **time-independent transition function of the Baker functions**.
- This constraint would be the basic of the integrable reduction framework in this talk

Sato–theoretic construction

- [Miwa shifts and lattice ASDYM equations]
- To the best of our knowledge, there appears to be relatively little research on discrete ASDYM equations.....
- Benincasa and Halburd [arXiv:1601.03102] employed a Bäcklund transformation to derive a discrete Yang equation, from which they obtained several integrable reductions.

If we can have a discrete analogue of these INPUT data, then the OUTPUT data should be corresponding lattice equations. This should be a good way to generate discrete integrable systems

SL(2) gauge group

$$\partial_n K = -(\partial_{n+1} J) J^{-1}, \quad \partial_n := \frac{\partial}{\partial t_n}, \quad n \geq 0,$$

$$\partial_0 K = \left[K, \frac{1}{2} \sigma_3 \right], \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

INPUT

[LMZ2026]

<https://arxiv.org/abs/2603.23060>

- The AKNS hierarchy (2.13): $(r, q) = (K_{21}, -K_{12})$.
- The GI hierarchy (2.18): $(p, q) = (J_{21} J_{11}^{-1}, -K_{12})$.
- The gauge factor between DNLS hierarchies (2.25): $s = J_{11}$.
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- The KP hierarchy (2.41): $u = K_{11,x}$.
- The mKP hierarchy (2.50): $v = -(\ln J_{11})_x$.

OUTPUT

Sato–theoretic construction

- [Miwa shifts and lattice ASDYM equations]
- To begin with, we assume C to be

$$C = \Pi := \begin{pmatrix} I_r & 0 \\ 0 & 0_{N-r} \end{pmatrix}, \quad 1 \leq r \leq N-1, \quad C^2 = C.$$

- Then introduce the Miwa shift

For later use, we adopt the abbreviations $\tilde{X} := \mathbb{T}_h X$ and $\Delta_h X := (\tilde{X} - X)/h$.

$$\mathbb{T}_h := \exp \left(\sum_{n \geq 1} \frac{h^n}{n} \partial_n \right), \quad \mathbb{T}_h X(t) = X(t + [h]), \quad [h] := \left(0, h, \frac{h^2}{2}, \frac{h^3}{3}, \dots \right).$$

- The exponential function satisfies

$$\tilde{E}(t, \lambda) = E(t, \lambda)(I_N - h\lambda C).$$

Sato-theoretic construction

- [Miwa shifts and lattice ASDYM equations]

Theorem 3.9 (Miwa-shift-generated discrete analogue). *Assume the hypotheses of Theorems 3.1 and 3.6 hold. Then there exists a unique λ -independent matrix U_h such that*

$$\tilde{\Psi}_a(\lambda) = L_h(\lambda)\Psi_a(\lambda), \quad L_h(\lambda) := I_N - h(\lambda C + U_h), \quad a \in \{0, \infty\}. \quad (3.22)$$

Equivalently,

$$\Delta_h W_a(\lambda) = \lambda(\tilde{W}_a(\lambda)C - CW_a(\lambda)) - U_h W_a(\lambda), \quad (3.23)$$

and

$$U_h = \tilde{K}C - CK = -(\Delta_h J)J^{-1}. \quad (3.24)$$

discrete analogue

$$\partial_n K = -(\partial_{n+1} J)J^{-1}, \quad \partial_n := \frac{\partial}{\partial t_n}, \quad n \geq 0,$$

Sato-theoretic construction

- [Miwa shifts and lattice ASDYM equations]

Remark 3.10. Introduce a second Miwa shift $\hat{X} := \mathbb{T}_k X$ and the associated difference operator $\Delta_k X := (\hat{X} - X)/k$. We then consider the pair of linear systems

$$\tilde{\Psi}_a(\lambda) = L_h(\lambda)\Psi_a(\lambda), \quad \hat{\Psi}_a(\lambda) = L_k(\lambda)\Psi_a(\lambda).$$

The compatibility $\hat{\tilde{\Psi}}_a = \tilde{\hat{\Psi}}_a$ then yields $\hat{L}_h(\lambda)L_k(\lambda) = \tilde{L}_k(\lambda)L_h(\lambda)$. Substituting the corresponding matrix expressions for the discrete gauge potentials gives the lattice Yang equation (cf. (3.6))

$$C(\Delta_h J)J^{-1} - C(\Delta_k J)J^{-1} + (\Delta_k \tilde{J})\tilde{J}^{-1}C - (\Delta_h \hat{J})\hat{J}^{-1}C = 0, \quad (3.25)$$

and the lattice LMP equation (cf. (3.7))

discrete Yang equation [BR2016]

$$\begin{aligned} \Delta_h(\hat{K}C - CK) - \Delta_k(\tilde{K}C - CK) \\ + (\hat{K}C - C\hat{K})(\hat{K}C - CK) - (\tilde{K}C - C\tilde{K})(\tilde{K}C - CK) = 0. \end{aligned} \quad (3.26)$$

Both may be regarded as discrete analogues of the ASDYM equations.

When $C = \text{diag}(1, -1)$, this system is equivalent to the discrete AKNS system via the Cauchy matrix approach [Zhao2016]

Sato–theoretic construction

- [Miwa shifts and lattice ASDYM equations]

Likewise, there will be a discrete coordinate flow

Theorem 3.11 (Discrete coordinate flow). *The relative coordinates satisfy*

$$\Delta_h w_{ij} - \tilde{w}_{i,j+1} C + C w_{i+1,j} = \tilde{w}_{i0} C w_{0j}, \quad (i, j) \in \mathbb{Z}^2. \quad (3.27)$$

Proof. For $\alpha \in \{\eta, \zeta\}$, set $R(\alpha) := I_N - h\alpha C$. Equation (3.23) is equivalent to

$$R(\eta) W_a(\eta)^{-1} = \tilde{W}_a(\eta)^{-1} L_h(\eta).$$

Using $L_h(\eta) - L_h(\zeta) = h(\zeta - \eta)C$, we obtain

$$R(\eta) W_{ab}(\eta, \zeta) - \tilde{W}_{ab}(\eta, \zeta) R(\zeta) = h(\tilde{W}_a(\eta)^{-1} C W_b(\zeta) - C).$$

Rearranging gives

$$\Delta_h W_{ab} - \zeta \tilde{W}_{ab} C + C W_{ab} \eta = C - \tilde{W}_a(\eta)^{-1} C W_b(\zeta).$$

Thus, a substitution of (2.3), (2.4), and (2.7), followed by sectorwise coefficient comparison, finally proves (3.27). $\square$

This is the discrete analogue of the Miura relation between P and Q

Likewise, the shift relation (3.27) is a discrete analogue of (3.8). Setting $(i, j) = (-1, -1)$ in (3.27) therefore gives

$$\Delta_h Q = C - \tilde{P} C P^{-1} = C - \tilde{J}^{-1} C J, \quad (3.28)$$

which serves as the discrete analogue of (3.10).

Integrable reductions

- [Assumptions]
- Previous discussions hold for all gauge groups. Here, for simplicity, we work on the $\mathrm{SL}(2)$ gauge group, which implies

$$\det J = 1, \quad \mathrm{tr} K = 0, \quad \det P = 1, \quad \mathrm{tr} Q = 0.$$

Condition (3.21) then gives $C = \Pi = \mathrm{diag}(1, 0)$. Thus for any 2×2 matrix $X \in \{J, K, P, Q\}$,

$$\partial_0 X = [X, \Pi] = \begin{pmatrix} 0 & -X_{12} \\ X_{21} & 0 \end{pmatrix}. \quad (4.2)$$

For later use, set $x := t_1$ and write

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & -K_{11} \end{pmatrix} =: \begin{pmatrix} u & -q \\ r & -u \end{pmatrix}, \quad J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} =: \begin{pmatrix} s & J_{12} \\ ps & J_{22} \end{pmatrix}. \quad (4.3)$$

Integrable reductions

- [Reduction to positive AKNS hierarchy]

Direct computation
without difficulty!
(see also in [LMZ2026])

Recalling the LMP equation (3.7) and setting $m = 1$, we obtain

$$\partial_{n-1}\partial_1 K - \partial_n[K, \Pi] - [\partial_{n-1}K, [K, \Pi]] = 0, \quad n \geq 1.$$

Expanding this equation shows that the entries satisfy

$$u_{x,t_{n-1}} = -(qr)_{t_{n-1}}, \quad q_{x,t_{n-1}} = -q_{t_n} - 2u_{t_{n-1}}q, \quad r_{x,t_{n-1}} = r_{t_n} - 2u_{t_{n-1}}r.$$

Rearranging these relations yields

$$q_{t_n} = -q_{x,t_{n-1}} + 2q\partial_x^{-1}(rq)_{t_{n-1}}, \quad r_{t_n} = r_{x,t_{n-1}} - 2r\partial_x^{-1}(qr)_{t_{n-1}}, \quad (4.4)$$

where $q_{t_0} = -q$ and $r_{t_0} = r$. This is precisely the positive AKNS hierarchy. The first nontrivial member of (4.4) is the coupled nonlinear Schrödinger (NLS) system

$$r_{t_2} = r_{xx} - 2qr^2, \quad q_{t_2} = -q_{xx} + 2q^2r. \quad (4.5)$$

Integrable reductions

In the case of $n = 1$, equations (2.17), (2.18) can be written explicitly as

- [A Sato–theoretic perspective]

$$\frac{\partial}{\partial x_1} \Psi(\lambda) = (\lambda Q + w_1 Q - Q w_1) \Psi(\lambda), \quad (2.20)$$

$$\frac{\partial}{\partial y_1} \Psi(\lambda) = \left(\lambda \frac{\partial}{\partial y_0} - \frac{\partial w_1}{\partial y_0} \right) \Psi(\lambda). \quad (2.21)$$

In this subsection, we restrict our discussion to continuous integrable systems; the discrete case can be treated in a similar manner. The AKNS variables arise naturally from the local expansion of the dressing matrix at $\lambda = \infty$. Recalling (3.20) with $a = \infty$, we obtain

$$\partial_n \Psi_\infty = -(\lambda^n \Pi + B_n(\lambda)) \Psi_\infty, \quad B_n := \sum_{i=1}^n \lambda^{n-i} \partial_{i-1} K. \quad (4.17)$$

Defining $\Phi^{[\text{AKNS}]} := e^{\xi/2} \Psi_\infty$ transforms (4.17) into the linear system for the AKNS hierarchy

$$\Phi_{t_n}^{[\text{AKNS}]} = - \left(\lambda^n \frac{\sigma_3}{2} + B_n(\lambda) \right) \Phi^{[\text{AKNS}]}. \quad (4.18)$$

Taking $n = 1$ yields the spectral problem for the AKNS hierarchy

$$\Phi_x^{[\text{AKNS}]} = - \left(\lambda \frac{\sigma_3}{2} + [K, \Pi] \right) \Phi^{[\text{AKNS}]} = - \frac{1}{2} \begin{pmatrix} \lambda & 2q \\ 2r & -\lambda \end{pmatrix} \Phi^{[\text{AKNS}]}. \quad (4.19)$$

This construction is compatible with [KIT2002], which is developed from the 2–component KP hierarchy

Integrable reductions

- [Reduction to positive GI hierarchy]

We next derive the GI hierarchy through the Miura transformation encoded by the J -matrix. Equation (3.4) with $n = 1$ expands as

$$\begin{pmatrix} s_x & J_{12,x} \\ (ps)_x & J_{22,x} \end{pmatrix} = \begin{pmatrix} -pqs & -qJ_{22} \\ -rs & -rJ_{12} \end{pmatrix},$$

whose first column implies

$$s_x = -pqs, \quad p_x s + ps_x = -rs.$$

On a domain where $s \neq 0$, eliminating s from the above relations gives

$$p_x = -r + p^2 q, \tag{4.6}$$

which is the Miura transformation between the AKNS and GI hierarchies [17]. Substituting (4.6) into (4.4) yields the positive GI hierarchy

$$p_{t_n} = p_{x,t_{n-1}} - 2p\partial_x^{-1}(pq_x + (pq)^2)_{t_{n-1}}, \quad p_{t_0} = p, \tag{4.7a}$$

$$q_{t_n} = -q_{x,t_{n-1}} - 2q\partial_x^{-1}(p_x q - (pq)^2)_{t_{n-1}}, \quad q_{t_0} = -q. \tag{4.7b}$$

The first nontrivial flow is given by

$$p_{t_2} = p_{xx} - 2p^2 q_x - 2p^3 q^2, \quad q_{t_2} = -q_{xx} - 2p_x q^2 + 2p^2 q^3. \tag{4.8}$$

Direct computation

without difficulty!

(see also in [LMZ2026])

Integrable reductions

- [Reduction to lattice AKNS system]
- In the continuous case, we start from the LMP equation. Thus for the discrete case, we will consider the discrete LMP equation

$$\Delta_h(\hat{K}C - CK) - \Delta_k(\tilde{K}C - CK) + (\hat{\tilde{K}}C - C\hat{K})(\hat{K}C - CK) - (\tilde{\hat{K}}C - C\tilde{K})(\tilde{K}C - CK) = 0.$$

- A substitution of K gives

$$\left(\frac{1}{h} - \frac{1}{k}\right) (\hat{\tilde{u}} - \hat{u} - \tilde{u} + u) + (\hat{\tilde{u}} - \hat{u})(\hat{u} - u) - (\hat{\tilde{u}} - \tilde{u})(\tilde{u} - u) + \hat{q}\hat{r} - \tilde{q}\tilde{r} = 0,$$

$$\frac{\tilde{q} - q}{h} - \frac{\hat{q} - q}{k} + (\tilde{u} - \hat{u})q = 0, \quad \frac{\hat{\tilde{r}} - \hat{r}}{h} - \frac{\tilde{\hat{r}} - \tilde{r}}{k} + \hat{\tilde{r}}(\hat{u} - \tilde{u}) = 0.$$

Integrable reductions

- [Reduction to lattice AKNS system]
- On the other hand, shift on J implies

$$\tilde{J} = \begin{pmatrix} 1 - h(\tilde{u} - u) & -hq \\ -h\tilde{r} & 1 \end{pmatrix} J.$$

Since $\det J = 1$ is preserved under Miwa shifts, relation (4.9) then implies

$$\det \begin{pmatrix} 1 - h(\tilde{u} - u) & -hq \\ -h\tilde{r} & 1 \end{pmatrix} = 1 - h(\tilde{u} - u) - h^2 q \tilde{r} = 1.$$

Rearranging this relation gives

$$\Delta_h u = \frac{\tilde{u} - u}{h} = -q\tilde{r}.$$

The shift relation of u implies an equation of (q,r)

$$\frac{\tilde{q} - q}{h} - \frac{\hat{q} - q}{k} + (\tilde{u} - \hat{u})q = 0,$$

Thus, finally we have:

$$\frac{\tilde{q} - q}{h} - \frac{\hat{q} - q}{k} = q^2(h\tilde{r} - k\hat{r}), \quad \frac{\hat{\tilde{r}} - \hat{r}}{h} - \frac{\hat{\tilde{r}} - \tilde{r}}{k} = \hat{\tilde{r}}^2(k\tilde{q} - h\hat{q}).$$

Integrable reductions

- [Reduction to lattice GI system]
- Similarly, we have to find a discrete version of the Miura transformation. Then, from the lattice AKNS system, we can get the lattice GI equation likewise.

We next use the discrete Miura relation to obtain the lattice GI system. With (4.3) and (4.10), we rewrite (4.9) as

$$\begin{pmatrix} \tilde{s} & \tilde{J}_{12} \\ \tilde{p}\tilde{s} & \tilde{J}_{22} \end{pmatrix} = \begin{pmatrix} 1 + h^2 q \tilde{r} & -h q \\ -h \tilde{r} & 1 \end{pmatrix} \begin{pmatrix} s & J_{12} \\ p s & J_{22} \end{pmatrix}.$$

This is the shift of J

The first column yields

$$\tilde{s} = (1 - hpq + h^2 q \tilde{r}) s, \quad \tilde{p}\tilde{s} = (p - h \tilde{r}) s.$$

Since $(1 - hpq + h^2 q \tilde{r})(1 + h q \tilde{p}) = 1$, it follows that

$$\frac{\tilde{s}}{s} = \frac{1}{1 + h q \tilde{p}}, \quad h \tilde{r} = p - \frac{\tilde{p}}{1 + h q \tilde{p}}. \quad (4.13)$$

Continuous Miura relation

$$p_x = -r + p^2 q,$$

This is the discrete Miura relation between the lattice AKNS and GI systems. A rearrangement of (4.13) implies

$$\Delta_h p = -\tilde{r} + p q \tilde{p} - h q \tilde{p} \tilde{r},$$

which is a discrete analogue of (4.6). Taking the continuous limit $h \rightarrow 0$ recovers (4.6).

Integrable reductions

- [Reduction to lattice GI system]

$$\frac{\tilde{q} - q}{h} - \frac{\hat{q} - q}{k} = q^2 (h\tilde{r} - k\hat{r}), \quad \frac{\hat{\tilde{r}} - \hat{r}}{h} - \frac{\tilde{\hat{r}} - \tilde{r}}{k} = \hat{\tilde{r}}^2 (k\tilde{q} - h\hat{q}).$$

$$\frac{\tilde{s}}{s} = \frac{1}{1 + hq\tilde{p}}, \quad h\tilde{r} = p - \frac{\tilde{p}}{1 + hq\tilde{p}}.$$

$$\frac{\hat{s}}{s} = \frac{1}{1 + kq\hat{p}}, \quad k\hat{r} = p - \frac{\hat{p}}{1 + kq\hat{p}}.$$



$$\frac{\tilde{q} - q}{h} - \frac{\hat{q} - q}{k} = q^2 \left(\frac{\hat{p}}{1 + kq\hat{p}} - \frac{\tilde{p}}{1 + hq\tilde{p}} \right).$$

Integrable reductions

- [Reduction to lattice GI system]

On the other hand, shifting the second relation in (4.13) in the k -direction and the second relation in (4.14) in the h -direction gives

$$h\hat{\tilde{r}} = \hat{p} - \frac{\hat{\tilde{p}}}{1 + h\hat{q}\hat{\tilde{p}}}, \quad k\hat{\tilde{r}} = \tilde{p} - \frac{\hat{\tilde{p}}}{1 + k\tilde{q}\hat{\tilde{p}}}.$$

These two relations respectively give

$$\frac{\hat{\tilde{p}} - \hat{p}}{h} = -\hat{\tilde{r}} + \frac{1}{h} \left(\hat{\tilde{p}} - \frac{\hat{\tilde{p}}}{1 + h\hat{q}\hat{\tilde{p}}} \right) = -\hat{\tilde{r}} + \hat{\tilde{p}}^2 \frac{\hat{q}}{1 + h\hat{q}\hat{\tilde{p}}},$$

and

$$\frac{\hat{\tilde{p}} - \tilde{p}}{k} = -\hat{\tilde{r}} + \frac{1}{k} \left(\hat{\tilde{p}} - \frac{\hat{\tilde{p}}}{1 + k\tilde{q}\hat{\tilde{p}}} \right) = -\hat{\tilde{r}} + \hat{\tilde{p}}^2 \frac{\tilde{q}}{1 + k\tilde{q}\hat{\tilde{p}}}.$$

Their difference eliminates $\hat{\tilde{r}}$ and yields

$$\frac{\hat{\tilde{p}} - \hat{p}}{h} - \frac{\hat{\tilde{p}} - \tilde{p}}{k} = \hat{\tilde{p}}^2 \left(\frac{\hat{q}}{1 + h\hat{q}\hat{\tilde{p}}} - \frac{\tilde{q}}{1 + k\tilde{q}\hat{\tilde{p}}} \right). \quad (4.16)$$

The second equation of
lattice GI system

Integrable reductions

- [Correspondence with relative coordinates]

Integrable systems	variables in ASDYM matrices	associated relative coordinates
AKNS (4.4)/(4.12)	$(r, q) = (K_{21}, -K_{12})$	$(-(w_{00})_{21}, (w_{00})_{12})$
GI (4.7)/(4.15)(4.16)	$(p, q) = \left(\frac{J_{21}}{J_{11}}, -K_{12} \right)$	$\left(\frac{(I_2 + w_{0,-1})_{21}}{(I_2 + w_{0,-1})_{11}}, (w_{00})_{12} \right)$

Table 1: Correspondence between integrable systems and relative coordinates I.

- This correspondence implies that if there is an explicit realization of the relative coordinates, then the continuous and discrete AKNS/GI systems are solved.
- In fact, the **Cauchy matrix approach** is exactly one way to achieve this. That's why we can use the CMA to derive lot of integrable systems.

Integrable reductions

- [Correspondence with relative coordinates]

Integrable systems	variables in ASDYM matrices	associated relative coordinates
HF (4.20)	$S = J^{-1}\sigma_3 J$	$(I_2 - w_{-1,0})\sigma_3(I_2 + w_{0,-1})$
HF (4.20)	$S = \sigma_3 - 2Q_x$	$\sigma_3 - 2\partial_x w_{-1,-1}$

Sometimes, the Q-matrix
is also needed
Check detailed proof in
[LMZ2026]

Table 2: Correspondence between integrable systems and relative coordinates II.

Integrable systems	variables in ASDYM matrices	associated relative coordinates
GDNLS (4.21)	$(\mu, \nu) = \left(J_{21} J_{11}^{\gamma-1}, -\frac{K_{12}}{J_{11}^{\gamma}} \right)$	$\left((I_2 + w_{0,-1})_{21} (I_2 + w_{0,-1})_{11}^{\gamma-1}, \frac{(w_{00})_{12}}{(I_2 + w_{0,-1})_{11}^{\gamma}} \right)$
CLL (4.22)	$(\mu, \nu) = \left(J_{21}, -\frac{K_{12}}{J_{11}} \right)$	$\left((I_2 + w_{0,-1})_{21}, \frac{(w_{00})_{12}}{(I_2 + w_{0,-1})_{11}} \right)$
KN (4.23)	$(\mu, \nu) = \left(J_{21} J_{11}, -\frac{K_{12}}{J_{11}^2} \right)$	$\left((I_2 + w_{0,-1})_{21} (I_2 + w_{0,-1})_{11}, \frac{(w_{00})_{12}}{(I_2 + w_{0,-1})_{11}^2} \right)$
pKN (4.24)	$(f, g) = \left(Q_{21}, -\frac{J_{12}}{J_{11}} \right)$	$\left((w_{-1,-1})_{21}, -\frac{(I_2 + w_{0,-1})_{12}}{(I_2 + w_{0,-1})_{11}} \right)$

Table 3: Correspondence between integrable systems and relative coordinates III.

Concluding remarks

- [Coordinate extension]
- The usual affine coordinates $w_{\{ij\}}$ with $i,j \geq 0$ is extended to $w_{\{ij\}}$ with i,j in Z .
- [Dimensional reduction constraint]
- The Sato–theoretic origin of the t_0 –flow is clarified.
- [Relative coordinate representation]
- Several integrable systems are expressed explicitly in terms of relative coordinates.

Concluding remarks

- [Future investigation]
- Through out this talk, we choose $G=SL(2)$ for convenience, a more natural setting is $G=GL(N)$, which allows reductions to multi-component integrable systems.
- We only consider the ASDYM hierarchy with single time sequence, a more natural setting is two sequences of hierarchy times, which allows reduction to breaking soliton equations, e.g. 2+1d NLS, CBS, [calculation is almost the same...]
- Cauchy matrix provides Grammian-type solutions. There are Wronskian-type ASDYM solutions [HH2020,HH2022], it should likewise admit formulations in terms of the relative coordinates.